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ORIGINAL PAPER

A contemporary linear representation theory for ordinary differential equations: multilinear algebra in folded arrays (folarrs) perspective and its use in multidimensional case

Metin Demiralp · Emre Demiralp

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Abstract In this paper, we extend the framework describing the probabilistic evolution of explicit unidimensional ODEs, which is described in the companion of this paper, to multidimensional cases. We show that an infinite set of linear ODEs accompanied by an initial condition (represented with an infinite vector) can also be constructed for the multidimensional cases. The principles that underly the construction of the equations and the truncated approximants of the solutions are the same. The crucial addition of this paper is the use of multiindex, folded and unfolded vectors and matrices. Unlike our earlier work on folded arrays, which relied on probabilistic principles of construction, in this work, we use a purely mathematical approach for the construction of the multidimensional structures. We provide a procedural description of how such constructions can be made.

Keywords Dynamical systems · Probability · Expectation values · Ordinary differential equations · Linear algebra

1 Introduction

This is the second of two papers investigating the probabilistic evolution equations of the initial value problems of explicit ODEs. In the first paper [1], we provide a proof of the key underlying concept using a canonical unidimensional ODE. In the current

M. Demiralp (✉)
İstanbul Teknik Üniversitesi Bilişim Enstitüsü, 34469 Maslak, Istanbul, Turkey
e-mail: metin.demiralp@gmail.com

E. Demiralp
Department of Psychology, University of Michigan, 1012 East Hall, 530 Church Street, Ann Arbor, MI 41809-1043, USA
e-mail: emredemi@umich.edu

paper, we show the benefit of our approach in the case of multidimensional ODEs. Specifically, we investigate initial value problems which involve ODEs with multiple unknowns, which is much more likely to be representative of phenomena that arise in the dynamics investigated not only in chemistry but also in other physical, natural and social sciences. Hence, this paper also sustains and perpetuates our dedication to extend the mathematical rigor of theoretical chemistry to multiple fields.

The principles which motivate the approaches that form the core of this paper are rooted in some of our earlier discoveries [2–6]. The first author is rigorously involved in developing and enhancing this mathematical framework, while the second author is in charge of applying these approaches to fields of computer science, engineering, neuroscience and mathematical modeling of human mental functioning. It is important to note that, there is an eclectic and diverse selection of research papers on the topic of multidimensional arrays [7–23]. While we are excited about the enthusiasm of the research community regarding this topic, there are very specific reasons why we prefer reductive decompositions as described in [24–28] and the straightforward extension of ordinary linear algebraic entities, vectors and matrices. Accordingly, we will provide a basic summary of the intuition behind folded vectors (folvecs) and folded matrices (folmats) before explaining the novel approach introduced in this paper.

This paper is organised as follows. The second section is devoted to a detailed reintroduction of folded arrays (folvecs and folmats). In the third section, we show a multivariate Taylor series expansion which contains folded arrays. The fourth section, builds upon this framework and describes the construction of probabilistic evolution equations. The fifth section mathematically illustrates the spectral properties of the evolution folmat. Subsequently, the sixth section details the properties of the evolution operator and its utilization. The seventh and eighth sections elaborate the considerations necessary for cases where hypertriangularity is involved. The ninth section is about the use of space extension for purposes of hypertriangular conicalization and removal of singularity from descriptive functions. The tenth and final section contains the concluding remarks.

2 Folded arrays (folarrs) perspective for multilinear algebra

Most of the systems observable or measurable by humans has multiple variables that describe the underlying dynamics. Previously, researchers have used the term "Tensor" to label a representation that characterizes the multidimensionality of various systems. This term is commonly used in continuum mechanics especially in the contexts of material properties such as strength. It is important to note that tensors are not only multidimensional objects. In fact they are intrinsically geometric objects. Tensors may correspond to various multilinear arrays by being dependent on the basis set for the vector field on which the tensor is defined. Furthermore, tensors do not depend on the coordinate system of the geometry under consideration. Even though a tensor is represented by a multilinear array, we do not need to establish relationships between multilinear arrays as tensors may connect many multiindex arrays, depending on how its elements transform from one coordinate system to the others. In fact, tensors are not the only mathematical constructs with these general class of properties. Spinors, for example, are similar mathematical objects. Their difference from tensors is primarily their transformational properties.

In our framework, the use of intrinsic mathematical objects introduces a number of roadblocks without providing any advantage. As a matter of fact, all we need for our approach is an array with multiple indices. This allows the construction of a simple algebra based on linear algebra of ordinary arrays such as matrices and vectors. In order to better explain our intuition, it is worthwhile to briefly describe the functional role of vectors and matrices. Vectors are used to denote the positions in multi coordinate systems while matrices can be considered as position representing entities in the spaces spanned by themselves. Matrices characterize transformations between spaces spanned by vectors. We want to categorize all arrays which may have more than two indices by two different objects, generalized forms of vectors and matrices. This section is devoted to this purpose.

2.1 Folding, row and column index discrimination and transposition of folarrs

Let us consider the following vector denoted by $\mathbf{a}$, whose elements are assumed to be real valued for simplicity (it is trivial to expand this to cover complex valued entities),

$$\mathbf{a} \equiv [\, a_1 \; a_2 \; \ldots \; a_n \,]^T . \tag{1}$$

This is a linear ordering and we can convert it to a rectangular array by reordering the terms. If we denote two prime factors of n by n_1 and n_2, then we can write

$$\mathbf{b}_1 \equiv \begin{bmatrix} a_1 & a_{n_1+1} & \cdots & a_{(n_2-1)n_1+1} \\ a_2 & a_{n_1+2} & \cdots & a_{(n_2-1)n_1+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_1} & a_{2n_1} & \cdots & a_{n_1 n_2} \end{bmatrix} \tag{2}$$

or

$$\mathbf{b}_2 \equiv \begin{bmatrix} a_1 & a_2 & \cdots & a_{n_2} \\ a_{n_2+1} & a_{n_2+2} & \cdots & a_{2n_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n_1-1)n_2+1} & a_{(n_1-1)n_2+2} & \cdots & a_{n_1 n_2} \end{bmatrix} \tag{3}$$

where the two dimensional arrays $\mathbf{b}_1$ and $\mathbf{b}_2$ represent the same vector in different orderings of the underlying elements. The construction of $\mathbf{b}_1$ can be explained as follows: We partition the vector $\mathbf{a}$ to n_1 elements of sub vectors and identify the first subvector as the first column of the two dimensional array $\mathbf{b}_1$. The remaining columns of this two dimensional array are respectively identified by the remaining subvectors of the above partitioning, that is, the kth ($k = 1, 2, \ldots, n_2$) column of $\mathbf{b}_1$ is identified by the kth subvector coming from the above mentioned partitioning. At the end of this operation, all elements are conserved but the ordering is changed from one dimension to two dimensions. If we fold the vector $\mathbf{a}$ to n_1-element-pieces, then $\mathbf{b}_1$

can be obtained by ordering the folds from left to right. Therefore, the resulting two dimensional array, $\mathbf{b}_1$, can be called "folded vector". It is important to note that, folding is not the act of ordering the subvectors from left to right. The ordering direction is perpendicular to the display screen from front to rear. Hence we will consider that the folding of a vector is an ordering over vertical lists in depth.

$\mathbf{b}_1$ is constructed by column-wise positioning of the folds whereas $\mathbf{b}_2$ is different in two aspects: (1) the number of the elements in each fold is n_2 not n_1, (2) positioning is done row-wise (transposes of folds are located downward). In this sense, the folding is not unique. On the other hand, the values n_1 and n_2 can be chosen in more than one way depending on the value of n. This is also another source of nonuniqueness. We have assumed that $n = n_1 n_2$ which may not be necessary. We could choose n_1 and n_2 such that $n = n_1 n_2 + r$ where the remainder r should remain less than the minimum of n_1 and n_2. Then, the resulting array would be constructed in type $n_1 \times (n_2 + 1)$ not $n_1 \times n_2$ and there would be missing elements to completely fill the two dimensional array to be constructed. We choose to assume the missing elements are zero in order to avoid introducing unintended sources of information which can lead to various types of unintended flexibility in the context of optimization. As we mentioned in the last paragraph, it is better to consider the rows as being ordered in depth from front to rear. Hence we intend to use the folding action in depth, that is, perpendicular to the display screen without regarding its style. The most common folding is considered to be on vertical and horizontal objects, first one of which is for vectors while the second is related to the transposes of vectors.

The general term of $\mathbf{a}$ is a one index entity a_i $(i = 1, 2, \ldots, n)$ while the general terms of $\mathbf{b}_1$ and $\mathbf{b}_2$ can be denoted by $b^{(1)}_{i_1,i_2}$ $(i_1 = 1, 2, \ldots, n_1; \quad i_2 = 1, 2, \ldots, n_2)$ and $b^{(2)}_{i_1,i_2}$ $(i_1 = 1, 2, \ldots, n_2; \quad i_2 = 1, 2, \ldots, n_1)$, respectively (here comma is used to emphasize the indices). We call the case of folding resulting in $\mathbf{b}_1$ "Canonical Folding" because of its natural characteristics. The reverse procedure to get $\mathbf{a}$ from $\mathbf{b}_1$ can be called "Unfolding". In this work we are going to deal with folded entities and will occasionally mention unfoldings.

On the other hand, we could increase the dimension to values greater than two in folding and the general term of the resulting array could contain indices as many as the dimensionality. In this sense, we interpret an array whose general term is denoted by $a_{i_1,\ldots,i_m}$ an m-dimensionally folded vector and symbolize it as $\mathbf{a}^{(m)}$ where the superscript shows the index dimension or the number of indices. We call $\mathbf{a}^{(m)}$ "folvec" to embody its folded vector nature.

Vectors can be considered as the specific forms of a more general entity, the matrix. They are in fact one column matrices and are known as the basic agents of the Cartesian spaces. Here we understand that a vector is one column matrix and its transpose is a not separate but adjoint entity. The multiplication operation defined on matrices produces matrices although the types of the operands are restricted due to certain compatibility conditions. This gives them mapping capabilities. In other words they can be considered as operators, in contrast to vectors which do not have such features. As a matter of fact, the general elements of matrices can be given through two index entities where one index is used as the summation agent of the multiplication while the other takes role in the identification of the resulting entity. This transformational property distinguishes the matrices from the vectors. This is the reason why we have not used the term “matrix” and we prefer to use the term “two dimensional array”. In the light of these discussions we may distinguish the multiindex arrays with respect to the roles of their indices.

If a multiindex array is multiplied such that certain portions of its indices are used as the summation agents, then that array can be considered as an operator transforming from a class of multiindex arrays to another type multiindex arrays. In that case the indices taking role in the sums of the multiplication can be considered as the column indices while the remaining indices are somehow identifiers or row indices. We first write the row indices in a comma separated format then we put a semicolon to distinguish the two type indices. The remaining indices follow the semicolon and are provided in a comma separated format. So the general term can be denoted as $a_{i_1,\ldots,i_m;j_1,\ldots,j_n}$ and the corresponding entity can be denoted by the shorthand notation $\mathbf{a}^{(m,n)}$ where the most important agents, the numbers of the row and column indices, m and n are emphasized through the superscript. We can consider $\mathbf{a}^{(m,n)}$ a folded matrix since it can be obtained from an ordinary matrix via foldings over rows and columns. Even though we do not get into the details of this procedure we call $\mathbf{a}^{(m,n)}$ “(m, n) Dimensional Folded Matrix” or briefly just “Folmat”. Then m and n will be called “Rowfolding Dimension” and “Columnfolding Dimension” respectively. The row indices $i_1, \ldots, i_m$ and column indices $j_1, \ldots, j_n$ can take values from certain separate or same subsets of integer set. Here, in this work, we will need to use a common finite set composed of some number of first members in the positive integer set.

As we have mentioned above, folvecs can be considered as the particular cases of folmats, similar to the ordinary linear algebraic entities, vectors and matrices. On the other hand, we have emphasized the importance of distinguishing the row indices and column indices because of the roles they play in mappings. We have introduced the utilization of the semicolon to emphasize on this discrimination. This urges us to utilize the semicolon in folvec general terms also. From now on we are going to write the general term of a folvec as $a_{i_1,\ldots,i_m;}$ and use the shorthand notation $\mathbf{a}^{(m,0)}$ for this entity.

We can now define the transpose of folvecs and folmats by following the logic used in ordinary linear algebra. We can write

$$\mathbf{a}^{(m,0)^T} \equiv \mathbf{b}^{(0,m)}, \quad b_{i_1,\ldots,i_m} \equiv a_{i_1,\ldots,i_m}, \quad m = 1, 2, 3, \ldots \tag{4}$$

which gives the vectors and their transposes of ordinary linear algebra for $m = 1$. This particular definition can be extended to all folmats as follows

$$\mathbf{A}^{(m,n)^T} \equiv \mathbf{B}^{(n,m)}, \quad B_{i_1,\ldots,i_n;j_1,\ldots,j_m} \equiv A_{j_1,\ldots,j_m;i_1,\ldots,i_n}, \quad m, n = 1, 2, 3, \ldots \tag{5}$$

where we have used capital letters to distinguish the folmats from folvecs.

We will use the word “folarr” as an abbreviation for “folded arrays”to refer to the category of folvecs and folmats.

2.2 Nested frames representation of folarrs

Folding in depth becomes increasingly nontrivial as the number of the indices increases. The picture formed after multiindex folding can be projected on a plane to facilitate understanding the geometry of regrouping and the computational aspects of the utilization of the folarrs. To this end we can use the concept of block matrix elements of matrices in ordinary linear algebra. We have not used subindices in the shorthand notation of the folmats above for brevity. However, the use of subindices can simplify the representation of multidimensional folmats. Let us consider a folmat denoted by $\mathbf{A}^{(m,n)}$ whose general term is $A_{i_1,\ldots,i_m;j_1,\ldots,j_n}$. If we focus on the elements where i_1 and j_1 take two specific values, then they form a folmat we denote by $\mathbf{A}_{i_1;j_1}^{(m-1,n-1)}$ whose general term is $A_{i_1,\ldots,i_m;j_1,\ldots,j_n}$ for $i_k = 1, 2, \ldots m_k$ and $j_\ell = 1, 2, \ldots n_\ell$ ($k = 2, 3, \ldots, m$ and $\ell = 2, 3, \ldots, n$). As can immediately be noticed, the only differences are the domains of the indices; two of the indices, i_1 and j_1 have one element domains hence they are just fixed. So we can write

$$\mathbf{A}^{(m,n)} \equiv \begin{bmatrix} \mathbf{A}_{1;1}^{(m-1,n-1)} & \ldots & \mathbf{A}_{1;n_1}^{(m-1,n-1)} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{m_1;1}^{(m-1,n-1)} & \ldots & \mathbf{A}_{m_1;n_1}^{(m-1,n-1)} \end{bmatrix} \tag{6}$$

which expresses a folmat in terms of folmats whose rowfold and columnfold dimensions are one less than the considered folmat's. We have used subfolmats whose block addresses are given by two subindices one for row and one for column. Each subfolmat here can also be represented through a framewise representation (rectangular array of subfolmats) just as in (6)

$$\mathbf{A}_{i_1;j_1}^{(m-1,n-1)} \equiv \begin{bmatrix} \mathbf{A}_{i_1,1;j_1,1}^{(m-2,n-2)} & \ldots & \mathbf{A}_{i_1,1;j_1,n_2}^{(m-2,n-2)} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{i_1,m_2;j_1,1}^{(m-2,n-2)} & \ldots & \mathbf{A}_{i_1,m_2;j_1,n_2}^{(m-2,n-2)} \end{bmatrix} \tag{7}$$

This can be generalized by the following recursion to reveal a nested frame structure

$$\mathbf{A}_{i_1,\ldots,i_k;j_1,\ldots,j_k}^{(m-k,n-k)} \equiv \begin{bmatrix} \mathbf{A}_{i_1,\ldots,i_k,1;j_1,\ldots,j_k,1}^{(m-k-1,n-k-1)} & \ldots & \mathbf{A}_{i_1,\ldots,i_k,1;j_1,\ldots,j_k,n_{k+1}}^{(m-k-1,n-k-1)} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{i_1,\ldots,i_k,m_{k+1};j_1,\ldots,j_k,1}^{(m-k-1,n-k-1)} & \ldots & \mathbf{A}_{i_1,\ldots,i_k,m_{k+1};j_1,\ldots,j_k,n_{k+1}}^{(m-k-1,n-k-1)} \end{bmatrix} \tag{8}$$

The consecutive utilization of (8) in (6) creates a nested rectangular frame structure. Hence all these representations can be gathered under a common name "Nested Frame Representations" of folmats. This representation facilitates not only multilinear array envisualization but also computer algebraic programming of issues related to the folarr algebra.

The recursion in (8) can be used until 0 appears in fold dimensionality in horizontal or vertical (or both) directions. Hence the value of k can not exceed $\min(m, n)$. If

$m = n$, the folmat under consideration is dimensionally symmetric, therefore we stop at $k = (m-1) = (n-1)$ where the subfolmats become the matrices of ordinary linear algebra. If m is not equal to n then either some folvecs or transposes of some folvecs become the stopping agents.

2.3 Multiplication of folmats

The multiplication of two folmats is defined as long as the column folding dimension of the first factor matches the rowfolding dimension of the second factor and indices can take values from certain compatible subsets of integers. We can write the following equalities to this end

$$\begin{aligned} \mathbf{C}^{(\ell,m)} &\equiv \mathbf{A}^{(\ell,p)}\mathbf{B}^{(p,m)}, \\ C_{i_1,\ldots,i_\ell;j_1,\ldots,j_m} &\equiv \sum_{\{k\}} A_{i_1,\ldots,i_\ell;k_1,\ldots,k_p} B_{k_1,\ldots,k_p;j_1,\ldots,j_m} \end{aligned} \tag{9}$$

where the sum is taken over all possible values of $k_1, k_2, \ldots, k_p$. In other words the above sum stands for p number of sums, limits of which are independent from those of other indices. In other words, summation is over an orthogonal hyperprismatic geometry. The action of a folmat on a folvec is defined accordingly and we can write the following definition identities

$$\begin{aligned} \mathbf{c}^{(\ell,0)} &\equiv \mathbf{A}^{(\ell,m)}\mathbf{b}^{(m,0)}, \\ c_{i_1,\ldots,i_\ell} &\equiv \sum_{\{j\}} A_{i_1,\ldots,i_\ell;j_1,\ldots,j_m} b_{j_1,\ldots,j_m}; \end{aligned} \tag{10}$$

where the summation is done same as above.

2.4 Outer product

We can define the outer product of two folvecs as follows

$$\begin{aligned} \mathbf{c}^{(\ell+m,0)} &\equiv \mathbf{a}^{(\ell,0)} \otimes \mathbf{b}^{(m,0)}, \\ c_{i_1,\ldots,i_\ell,j_1,\ldots,j_m;} &\equiv a_{i_1,\ldots,i_\ell;} b_{j_1,\ldots,j_m;} \end{aligned} \tag{11}$$

which is not necessarily a commutative operation. This operation produces new folvecs from existing folvecs such that the resulting folvecs' rowfolding dimension is the sum of the rowfolding dimensions of the operands. Since the columnfolding dimension is zero for both operands and the resulting columnfolding dimension remains same, zero, we can generalize this bilinear operation to folmats such that the resulting folmat's rowfolding and columnfolding dimensions are equal to the separate sums of rowfolding and column folding dimensions of the operands. Therefore the outer product of two folmats can be defined as follows

$$\mathbf{C}^{(m_1+m_2,n_1+n_2)} \equiv \mathbf{A}^{(m_1,n_1)} \otimes \mathbf{B}^{(m_2,n_2)},$$
$$C_{i_1,\ldots,i_{m_1},k_1,\ldots,k_{m_2};j_1,\ldots,j_{n_1},\ell_1,\ldots,\ell_{n_2}} \equiv C_{i_1,\ldots,i_{m_1};j_1,\ldots,j_{n_1}} B_{k_1,\ldots,k_{m_2};\ell_1,\ldots,\ell_{n_2}} \qquad (12)$$

These bilinear operations have been called "Outer Product " because they produce higher dimensional entities from the operands (as a matter of fact, this higher dimension is the sum of the dimensions of the operands) in folding dimensions. So they are compatible with the tensor product definition in the scientific literature.

We can also define the outer power of a folvec as follows

$$\mathbf{a}^{(m,0)^{\otimes n}} \equiv \bigotimes_{i=1}^{n} \mathbf{a}^{(m,0)} \equiv \mathbf{a}^{(m,0)} \otimes \cdots \otimes \mathbf{a}^{(m,0)} \qquad (13)$$

whose particular case is the outer power of a vector of ordinary linear algebra. If we write $\mathbf{a}$ for $\mathbf{a}^{(m,0)}$ then we obtain

$$\mathbf{a}^{\otimes n} \equiv \bigotimes_{i=1}^{n} \mathbf{a} \equiv \mathbf{a} \otimes \cdots \otimes \mathbf{a}. \qquad (14)$$

which is a supersymmetric folvec whose general term is $a_{i_1} \ldots a_{i_n}$. The case where $n = 0$ is assumed to produce a folvec of zero dimension and general term involves no index. Hence $n = 0$ corresponds to a scalar which can be taken as 1 by following the conventions on the powering operation of standard algebra.

The outer products of the images under folmats obey the following rule

$$\left(\mathbf{A}^{(\ell_1,m_1)}\mathbf{b}^{(m_1,0)}\right) \otimes \left(\mathbf{C}^{(\ell_2,m_2)}\mathbf{d}^{(m_2,0)}\right) = \left(\mathbf{A}^{(\ell_1,m_1)} \otimes \mathbf{C}^{(\ell_1,m_1)}\right) \left(\mathbf{b}^{(m_1,0)} \otimes \mathbf{d}^{(m_2,0)}\right) \qquad (15)$$

A very important agent in the matrices of ordinary linear algebra is the unit or identity matrix which is actionless on its operands, it multiplies its operand by 1. Its outer powers play the same roles over their operand folvecs or folmats. We denote the identity matrix of ordinary linear algebra by $\mathbf{I}^{(1,1)}$ whose general term is the well known Kroenecker delta symbol $\delta_{i;j}$ which vanishes for all i, j couples except the ones of same numerical values where $\delta_{i;i}$ becomes 1. We can write

$$\mathbf{I}^{(m,m)} \equiv \bigotimes_{i=1}^{n} \mathbf{I}^{(1,1)} \qquad (16)$$

where the general element of the folded identity matrix can be given as follows

$$I_{i_1,\ldots,i_m;j_1,\ldots,j_m} = \prod_{k=1}^{m} \delta_{i_k;j_k}. \qquad (17)$$

The i or j indices here can take values from different integer sets however, we assume that the domains of i_k and j_k are same for each individual k value. It is not hard to show that $\mathbf{I}^{(m,m)}$ leaves its operand unchanged after premultiplication.

2.5 Inner product, norm, trace, determinant

If any two folmats have the same rowfolding and columnfolding dimensions and their indices have the same domains, then one can define inner product between them by following the Frobenius norm definition over the matrices of ordinary linear algebra. We can write

$$\left(\mathbf{A}^{(m,n)}, \mathbf{B}^{(m,n)}\right) \equiv \sum_{\{i\},\{j\}} A_{i_1,\ldots,i_m;j_1,\ldots,j_n} B_{i_1,\ldots,i_m;j_1,\ldots,j_n} \tag{18}$$

which remains valid over the real-valued entities and satisfies the basic definition properties of the binary inner product functional mapping from folmats to the real values: (1) symmetry with respect to arguments, (2) linearity with respect to first argument, (3) positive valuedness when both arguments become identical by being nonzero.

The norm of any given folmat can be induced from this inner product definition as long as the considered folmat has real valued elements

$$\left\|\mathbf{A}^{(m,n)}\right\| \equiv \left(\mathbf{A}^{(m,n)}, \mathbf{A}^{(m,n)}\right)^{\frac{1}{2}} \tag{19}$$

Although we have not previously stated explicitly, we can call a folmat “square” when its rowfolding and columnfolding dimensions are equivalent. For a square folmat we can define its trace as follows

$$Tr\left(\mathbf{A}^{(m,m)}\right) \equiv \sum_{\{i\}} A_{i_1,\ldots,i_m;i_1,\ldots,i_n} \tag{20}$$

This implies

$$\left(\mathbf{A}^{(m,n)}, \mathbf{B}^{(m,n)}\right) \equiv Tr\left(\mathbf{A}^{(m,n)^T}\mathbf{B}^{(m,n)}\right) \equiv Tr\left(\mathbf{B}^{(m,n)^T}\mathbf{A}^{(m,n)}\right) \tag{21}$$

and

$$\left\|\mathbf{A}^{(m,n)}\right\| \equiv \left[Tr\left(\mathbf{A}^{(m,n)^T}\mathbf{A}^{(m,n)}\right)\right]^{\frac{1}{2}} \equiv \left[Tr\left(\mathbf{A}^{(m,n)}\mathbf{A}^{(m,n)^T}\right)\right]^{\frac{1}{2}}. \tag{22}$$

The determinant of a square folmat is defined in accordance with the corresponding entity in the ordinary linear algebra. Its evaluation can be realised by using the nested frame representation definition where a planar array, in other words a matrix structure, is obtained. In this framework, all properties of the determinants remain valid.

2.6 Differentiation of an outer power

The differentiation of the outer power of a parametric ordinary vector $\mathbf{a}(t)$ with respect to the parameter t satisfies the following equality

$$\frac{d\left[\mathbf{a}(t)^{\otimes m}\right]}{dt} = \sum_{\ell=0}^{m-1} \mathbf{a}(t)^{\otimes \ell} \otimes \dot{\mathbf{a}}(t) \otimes \mathbf{a}(t)^{\otimes(m-\ell-1)} \tag{23}$$

where dot stands for the differentiation with respect to t and zeroth outer power produces just 1 as a zero dimensional folvec, by definition as we have stated above.

3 Multivariable Taylor series

The Taylor series of a function $f(x_1, \ldots, x_n)$ is assumed to be analytic around a reference point $\left(x_1^{(r)}, \ldots, x_n^{(r)}\right)$ and therefore has unique derivatives in all possible orders, at the reference point can be expressed as follows

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{\infty} \mathbf{f}^{(0,i)} \mathbf{x}^{\otimes i} \tag{24}$$

where each additive term in the summand is a scalar since it is the image of the outer power of the variable vector $\mathbf{x}$ whose explicit structure is given below

$$\mathbf{x} \equiv \left[\left(x_1 - x_1^{(r)}\right) \; \ldots \; \left(x_n - x_n^{(r)}\right) \right]^T \tag{25}$$

under the folmat $\mathbf{f}^{(0,i)}$ whose general term has no row index. The general term of $\mathbf{f}^{(0,i)}$, which is denoted by $f^{(0,i)}_{;j_1,\ldots,j_i}$, can be given as follows

$$f^{(0,i)}_{;j_1,\ldots,j_i} \equiv \frac{1}{i!} \left. \frac{\partial^i f(x_1, \ldots, x_n)}{\partial x_{j_1} \ldots \partial x_{j_i}} \right|_{x_k = x_k^{(r)},\; k=1,\ldots,n}, \quad j_1, \ldots, j_i = 1, 2, \ldots, n \tag{26}$$

where the semicolon is deliberately shown to emphasize the nonexistence of the row indices which would be appearing at the left of the semicolon if they would exist and we have not considered the symmetry in the partial differentiation on purpose to avoid the complications coming from these simplifications in folmat notation.

These formulae can be extended to the Taylor series expansion of vector valued multivariate functions via following equalities

$$\mathbf{f}(x_1, \ldots, x_n) = \sum_{i=0}^{\infty} \mathbf{f}^{(1,i)} \mathbf{x}^{\otimes i} \tag{27}$$

where each term in the summand is a vector since it is the image of the outer power of the variable vector $\mathbf{x}$ which is given through (25) under the folmat $\mathbf{f}^{(1,i)}$ whose general term has just one row index. The general term of $\mathbf{f}^{(1,i)}$, which is denoted by $f^{(1,i)}_{j_1;k_1,\ldots,k_i}$, can be given as follows

$$f^{(1,i)}_{j_1;k_1,\ldots,k_i} \equiv \frac{1}{i!} \left. \frac{\partial^i f_{j_1}(x_1,\ldots,x_n)}{\partial x_{k_1}\ldots\partial x_{k_i}} \right|_{x_\ell = x_\ell^{(r)},\ \ell=1,\ldots,n}, \quad j_1, k_1, \ldots, k_n = 1, 2, \ldots, n \tag{28}$$

where $f_{j_1}(x_1,\ldots,x_n)$ stands for the j_1th element of the vector $\mathbf{f}(x_1,\ldots,x_n)$.

4 ODE set and probabilistic evolution equation

Let us now consider the following ODE set and the accompanying initial conditions

$$\dot{\xi}_i(t) = f_i(\xi_1(t),\ldots,\xi_n(t)); \quad \xi_i(0) = a_i, \quad i = 1, 2, \ldots, n \tag{29}$$

where we have assumed the autonomy without any loss of generality since autonomy can be always obtained by a simple space extension by considering the independent variable t as if it is an extra unknown everywhere it appears. The multivariate Taylor series expansions of f_is urge us to construct ODEs over the power basis set whose general element is given by the following power folvec

$$\mathbf{x}_i(t) \equiv \mathbf{x}(t)^{\otimes i}, \quad i = 0, 1, 2, \ldots \tag{30}$$

where

$$\mathbf{x}(t) \equiv \left[\left(\xi_1(t) - x_1^{(r)}\right) \ \ldots \ \left(\xi_n(t) - x_n^{(r)}\right) \right]^T. \tag{31}$$

The utilization of (31) and (27) in (29) enables us to write

$$\dot{\mathbf{x}}(t) = \sum_{i=0}^{\infty} \mathbf{f}^{(1,i)} \mathbf{x}(t)^{\otimes i} \tag{32}$$

which can be combined with the following equality

$$\frac{d\left[\mathbf{x}(t)^{\otimes i}\right]}{dt} = \sum_{\ell=0}^{i-1} \mathbf{x}(t)^{\otimes \ell} \otimes \dot{\mathbf{x}}(t) \otimes \mathbf{x}(t)^{\otimes (i-\ell-1)} \tag{33}$$

to write the following equations

$$\frac{d\left[\mathbf{x}_i(t)\right]}{dt} \equiv \frac{d\left[\mathbf{x}(t)^{\otimes i}\right]}{dt} = \sum_{\ell=0}^{i-1} \mathbf{x}(t)^{\otimes \ell} \otimes \left(\sum_{j=0}^{\infty} \mathbf{f}^{(1,j)} \mathbf{x}(t)^{\otimes j}\right) \otimes \mathbf{x}(t)^{\otimes(i-\ell-1)},$$

$$i = 0, 1, 2, \ldots \tag{34}$$

If $\mathbf{I}_n$ stands for the $n \times n$ type identity matrix, we can consider it as a folmat whose rowfolding and columnfolding dimensions are just 1. This permits us to use outer powers of a folmat in accordance with the definitions we have given previously. By using the outer power of identity matrix we can write

$$\mathbf{x}(t)^{\otimes \ell} = (\mathbf{I}_n \mathbf{x}(t))^{\otimes \ell} = \mathbf{I}_n^{\otimes \ell} \mathbf{x}(t)^{\otimes \ell}, \tag{35}$$

$$\mathbf{x}(t)^{\otimes \ell} \otimes \left(\mathbf{f}^{(1,j)} \mathbf{x}(t)^{\otimes j}\right) \otimes \mathbf{x}(t)^{\otimes(i-\ell-1)} = \left(\mathbf{I}_n^{\otimes \ell} \otimes \mathbf{f}^{(1,j)} \otimes \mathbf{I}_n^{\otimes(i-\ell-1)}\right) \mathbf{x}(t)^{\otimes(i+j-1)} \tag{36}$$

which allows us to rewrite (34) as follows

$$\begin{aligned}
\dot{\mathbf{x}}_i(t) &= \sum_{j=0}^{\infty} \left(\sum_{\ell=0}^{i-1} \mathbf{I}_n^{\otimes \ell} \otimes \mathbf{f}^{(1,j)} \otimes \mathbf{I}_n^{\otimes(i-\ell-1)}\right) \mathbf{x}(t)^{\otimes(i+j-1)} \\
&= \sum_{j=i-1}^{\infty} \left(\sum_{\ell=0}^{i-1} \mathbf{I}_n^{\otimes \ell} \otimes \mathbf{f}^{(1,j-i+1)} \otimes \mathbf{I}_n^{\otimes(i-\ell-1)}\right) \mathbf{x}(t)^{\otimes j} \\
&= \sum_{j=0}^{\infty} \mathbf{E}^{(i,j)} \mathbf{x}(t)^{\otimes j} \equiv \sum_{j=0}^{\infty} \mathbf{E}^{(i,j)} \mathbf{x}_j(t), \quad i = 0, 1, 2, \ldots
\end{aligned} \tag{37}$$

$$\mathbf{E}^{(i,j)} = \sum_{\ell=0}^{i-1} \mathbf{I}_n^{\otimes \ell} \otimes \mathbf{f}^{(1,j-i+1)} \otimes \mathbf{I}_n^{\otimes(i-\ell-1)}, \quad i, j = 0, 1, 2, \ldots \tag{38}$$

where $\mathbf{f}^{(1,j-i+1)}$ is assumed to be vanishing when j is less than $i-1$. We call $\mathbf{E}$, whose general folmat element (somehow block element) is $\mathbf{E}^{(i,j)}$, "Evolution Folmat" within an analogy to the previous companion paper. The evolution folmat here can be unfolded to get an upper block Hessenberg form and this unfolded form turns out to be block tridiagonal when the folmat $\mathbf{f}^{(1,0)}$, which is in fact an ordinary vector, vanishes. This vanishing property can be (almost) always provided in an extended space by using an appropriate space extension.

$\mathbf{x}_i(t)$ is a folvec having n^i elements while $\mathbf{E}^{(i,j)}$ stands for a folmat having n^i rows and n^j columns. Even though their folarr structures are more abstract than their ordinary linear algebraic counterparts, we prefer to use their unfolded forms through nested frame representation.

The original set of ODEs and accompanying initial conditions can be concisely expressed in the following form

$$\dot{\overline{\mathbf{x}}}(t) = \mathbf{E}\overline{\mathbf{x}}(t), \quad \overline{\mathbf{x}}(0) = \overline{\mathbf{a}} \tag{39}$$

where

$$\overline{\mathbf{x}}(t) \equiv [\mathbf{x}_0 \ \ldots \ \mathbf{x}_n \ \ldots \ ]^T, \quad \overline{\mathbf{a}} \equiv [\mathbf{a}_0 \ \ldots \ \mathbf{a}_n \ \ldots \ ]^T; \tag{40}$$

$$\mathbf{a}_i \equiv \mathbf{a}^{\otimes i}, \quad \mathbf{a} \equiv \left[\left(a_1 - x_1^{(r)} \right) \cdots \left(a_n - x_n^{(r)} \right) \right] \tag{41}$$

$$\mathbf{E} \equiv \begin{bmatrix} \mathbf{E}^{(0,0)} & \ldots & \mathbf{E}^{(0,n)} & \ldots \\ \vdots & \ddots & \vdots & \ldots \\ \mathbf{E}^{(n,0)} & \ldots & \mathbf{E}^{(n,n)} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{42}$$

(39) is an infinite linear folvec equation being equivalent to an infinite vector ODE via unfolding all folded arrays. The infinite folvecs $\overline{\mathbf{x}}(t)$ and $\overline{\mathbf{a}}$ have the general terms as the folvecs $\mathbf{x}(t)^{\otimes i}$ and $\mathbf{a}^{\otimes i}$ $(i = 0, 1, \ldots)$, respectively. So they are composed of different type elements from the same class entities. However, they can be considered vector blocks via unfolding operation as explained before.

The Evolution Folmat is in an infinite square structure and can be unfolded in both row and column index groups to get an infinite square matrix. Hence, its natural number powers are defined and the multiplication between folmats can be used to this end. The existence of this powering possibility also permits us to define the functions of the Evolution Folmat and we can use the exponential function to get the solution of (39) in the following form

$$\mathbf{x}(t) = e^{t\mathbf{E}}\mathbf{a} \tag{43}$$

The inspiration from the previous companion paper urges us to call the exponential folmat $e^{t\mathbf{E}}$ "Probabilistic Propagation Folmat" or simply "Folmat Propagator" due to the fact that it describes the propagation of the probabilistic behavior of the system without depending on the initial values and has universality in this sense.

5 Certain spectral properties of the Evolution Folmat for the hypertriangular case

The case where the folmat $\mathbf{f}^{(1,0)}$ vanishes, facilitates the analysis very much since the Evolution Folmat element $\mathbf{E}^{(i,j)}$ vanishes when i remains less than j. This corresponds to the somehow block upper triangular form. We call this triangularity "Upper Hypertriangularity" due to its folmat structure. Since the Evolution Folmat maps from the infinite folvecs to infinite folvecs with same cardinality (from same type entities to same type entities), it is possible to mention eigenvalues. We can write

$$\mathbf{E}\mathbf{e}^{(r)} = \epsilon\mathbf{e}^{(r)} \tag{44}$$

and

$$\mathbf{E}^{\dagger}\mathbf{e}^{(\ell)} = \epsilon\mathbf{e}^{(\ell)} \tag{45}$$

where the general element ((i, j)th) of the "Adjoint Evolution Folmat" corresponds to the transpose of the ordinary matrix which is unfolded form of $\mathbf{E}^{(j,i)}$. The Adjoint Evolution Folmat has lower hypertriangular form and hence the adjoint concept is equivalent to the transpose concept in the ordinary linear algebra.

The hypertriangular folmat structure above enables us to divide the eigenvalue determination of the infinite folmat $\mathbf{E}$ to a sequence of eigenvalue problems of finite folmats $\mathbf{E}^{(i,i)}$s ($i = 0, 1, 2, \ldots$). We can write these individual eigenvalue problems as follows

$$\mathbf{E}^{(i,i)}\mathbf{e}^{(r,i)} = \sum_{\ell=0}^{i-1} \mathbf{I}_n^{\otimes\ell} \otimes \mathbf{f}^{(1,1)} \otimes \mathbf{I}_n^{\otimes(i-\ell-1)}\mathbf{e}^{(r,i)} = \epsilon^{(i)}\mathbf{e}^{(r,i)}, \quad i = 0, 1, 2, \ldots \tag{46}$$

where the case where $i = 0$ corresponds to vanishing scalar value and $\mathbf{f}^{(1,1)}$ has the following explicit matrix form, whose elements are evaluated at the reference point,

$$\mathbf{f}^{(1,1)} \equiv \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)_{\mathbf{x}=\mathbf{x}^{(r)}} & \cdots & \left(\frac{\partial f_1}{\partial x_n}\right)_{\mathbf{x}=\mathbf{x}^{(r)}} \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial f_n}{\partial x_1}\right)_{\mathbf{x}=\mathbf{x}^{(r)}} & \cdots & \left(\frac{\partial f_n}{\partial x_n}\right)_{\mathbf{x}=\mathbf{x}^{(r)}} \end{bmatrix} \tag{47}$$

which is the Jacobian Matrix of f functions with respect to the independent variables, evaluated at $\mathbf{x} = \mathbf{x}^{(r)}$. Although it is a real valued and square type matrix, there is no warranty for its symmetry. Hence its spectrum does not need to be on the real axis of its eigenvalue complex plane. There is also no warranty for the equality between algebraic and geometric multiplicities of the multiple eigenvalues. This means that there is no warranty for the diagonalizability. However, it is always possible to convert it to a Jordan canonical form (in upper or lower format).

It is not hard to see that $\mathbf{E}^{(1,1)}$ matches $\mathbf{f}^{(1,1)}$. Therefore the eigenvalues and the eigenvectors of $\mathbf{f}^{(1,1)}$ are $\epsilon_1^{(1)}$, …, $\epsilon_n^{(1)}$ and $\mathbf{e}_1^{(r,1)}$, …, $\mathbf{e}_n^{(r,1)}$, respectively. This information can be used to find the eigenvalues and eigenvectors of $\mathbf{E}^{(i,i)}$ because of the direct product structure in $\mathbf{E}^{(i,i)}$. We can write

$$\epsilon_{\mathbf{k}}^{(i)} = \sum_{j=1}^{i} \epsilon_{k_j}^{(1)}, \quad \mathbf{e}_{\mathbf{k}}^{(r,i)} = \bigotimes_{j=1}^{i} \mathbf{e}_{k_j}^{(r,1)}, \quad 1 \le k_1, \ldots, k_i \le n \tag{48}$$

where $\mathbf{k}$ stands the set of k_js ($j = 1, 2, \ldots, i$). The superscript r in these formulae recalls the word "right". Hence the eigenvectors here are right eigenvectors. The left eigenvectors can be found similarly. To this end we can write

$$\mathbf{E}^{(i,i)^{\dagger}}\mathbf{e}^{(\ell,i)} = \sum_{\ell=0}^{i-1} \mathbf{I}_n^{\otimes \ell} \otimes \mathbf{f}^{(1,1)^T} \otimes \mathbf{I}_n^{\otimes(i-\ell-1)}\mathbf{e}^{(\ell,i)} = \epsilon^{(i)}\mathbf{e}^{(\ell,i)}, \quad i = 0, 1, 2, \ldots \tag{49}$$

$$\epsilon_{\mathbf{k}}^{(i)} = \sum_{j=1}^{i} \epsilon_{k_j}^{(1)}, \quad \mathbf{e}_{\mathbf{k}}^{(\ell,i)} = \bigotimes_{j=1}^{i} \mathbf{e}_{k_j}^{(\ell,1)}, \quad 1 \leq k_1, \ldots, k_i \leq n \tag{50}$$

This analysis can become peculiar in cases where all eigenvalues are different. However, it can be modified for the case of multiple eigenvalues. This can be done for cases where the algebraic and geometric multiplicities are equal or different. We do not intend to get into further details of this issue.

We can use the spectral entities for the spectral decomposition of the $\mathbf{E}^{(i,i)}$ and therefore for the evaluation of the exponential functions $\mathrm{e}^{t\mathbf{E}^{(i,i)}}$s for all natural number values of i. This fact enables us to evaluate the Folmat Propagator. We find this analysis sufficient for our purposes.

6 Evolution operator and its utilization

As we have shown in the companion [1] of this paper, the Evolution Matrix transpose somehow corresponds to the first order differential operator we had called "Evolution Operator". One can show that the same is true for the Evolution Folmat transpose. Even though we do not intend to give the intermediate details, the following first order partial differential operator corresponds to the transpose of Evolution Folmat

$$\mathcal{E}^{\dagger} \equiv \sum_{i=1}^{n} f_i\left(x_1, \ldots, x_n\right) \frac{\partial}{\partial x_i}. \tag{51}$$

The eigenvalues of this operator match the eigenvalues of the Evolution Folmat. The Taylor series coefficients of this Evolution Operator's eigenfunctions correspond to the left eigenvectors of the Evolution Folmat. Hence the eigenfunctions of this operator is one of the basic determining agents in the investigation of the truncation approximant convergence as in the case of one unknown case. However, the most important difference is the type of the differential equation giving the eigenfunction. Due to its structure the relevant equation is a partial differential equation. Therefore the solution of the eigenfunction equation may be nontrivial. Nevertheless the eigenvalues need not be determined since they are known from the Evolution Folmat.

The Evolution Operator, which is a partial differential operator in this work has the same features of a unidimensional ordinary differential operator. That is, its action on a product obeys the Leibnitz rule of differentiation on products. Similarly its action on a composite function is the image of its action on the inner function under the outer function. The Leibnitz rule underlies the following property: " The product of any two eigenfunctions of the evolution operator is also an eigenfunction corresponding to the sum of the eigenvalues corresponding to the eigenfunction factors". This means that the eigenvalues and eigenfunctions of the Jacobian of the descriptive functions in the set of ODEs under consideration play the role of generators to produce all eigenpairs of the evolution operator. The determination of the eigenfunction generators requires the solution of the corresponding partial differential equations. We do not intend to focus on this issue here.

7 Providing hypertriangularity

The cases where the Evolution Folmat becomes upper hypertriangular (block triangular) are very important since the eigenvalues can be separated to finitely many subsets and the exponential folmat evaluation becomes rather easy. As we have shown in this paper and in its companion the descriptive function vector $\mathbf{f}(x_1, \ldots, x_n)$ must vanish at the expansion point to provide the hypertriangularity in the evolution folmat structure. In other words, the folmat coefficient $\mathbf{f}^{(1,0)}$, which is apparently depending on the expansion point, should vanish. Even though we prefer to deal with entire descriptive functions, there is no warranty to have a vanishing descriptive function vector at the expansion point unless certain specific points exist and they are chosen as the expansion point. However, we can follow the approach of space extension as we did in the companion paper. To explain explicitly, we can write the equations

$$\dot{\xi}_i = f_i(\xi_1, \ldots, \xi_n), \quad i = 1, 2, \ldots, n \tag{52}$$

and define

$$\eta_i \equiv f_i(\xi_1, \ldots, \xi_n), \quad i = 1, 2, \ldots, n. \tag{53}$$

The resulting $2n$ equations are homogeneously linear in ηs in the descriptive functions. Hence the $2n$ dimensional subspace where ηs vanish provide the hypertriangularity if we choose the expansion point there. So this appropriate space extension provides us the hypertriangularity.

8 Technicality of hypertriangular conicality

If the descriptive functions are multinomials in their arguments with degrees not exceeding two, then we call them “conical”. If this is the case, then the nonvanishing folmats of the descriptive function vector $\mathbf{f}(x_1, \ldots, x_n)$, are just $\mathbf{f}^{(1,0)}$, $\mathbf{f}^{(1,1)}$, $\mathbf{f}^{(1,2)}$. If this multinomial vanishes at the expansion point then $\mathbf{f}^{(1,0)}$ also vanishes. Then the folmat elementwise probabilistic evolution equations can be given through the following equations

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}^{(1,1)}\mathbf{x}_i(t) + \mathbf{f}^{(1,2)}\mathbf{x}_{i+1}(t), \quad i = 1, 2, \ldots, \tag{54}$$

which can be truncated at $i = n$ to produce truncation approximants. In that case we need to impose $\mathbf{x}_{n+1}(t) \equiv \mathbf{0}$ to get true truncation. This imposition and the first order recursive nature of (54) enables us to determine the unknown entities, $\mathbf{x}(t)$s systematically and provides us many beautiful structures in the truncation approximants. What we have done for the one unknown case can be formally transformed to this case.

9 Space extension based hypertriangular conicalization and singularity removal in descriptive functions

As we have stated in this paper and in its companion, the space extension concept is a very efficient method for the restructuring of the given ODEs. It increases the number of the unknowns in infinitely many ways such that the desired structures can be attained in the descriptive functions of the resulting ODEs. This can be realized to get hypertriangular conicality in the descriptive functions in many cases. The singularities in the original forms of the descriptive functions can be removed but not annihilated. They appear in the resulting equations' accompanying initial conditions. To simply explain this issue, we can focus on the following initial value problem of one unknown ODE.

$$\dot{\xi}(t) = \ln\left(1 + \xi(t)\right), \quad \xi(0) = a \tag{55}$$

To extend the space of unknowns we can define

$$\begin{aligned} u_1(t) &\equiv \ln\left(1 + \xi(t)\right), \quad u_2(t) \equiv \frac{1}{1+\xi(t)}, \quad u_3(t) \equiv \frac{1}{(1+\xi(t))^2}, \\ u_4(t) &\equiv \frac{\ln\left(1+\xi(t)\right)}{1+\xi(t)}, \quad u_5(t) \equiv \ln\left(1 + \xi(t)\right)^2 \end{aligned} \tag{56}$$

which permits us to write the following ODEs

$$\begin{aligned} \dot{\xi} &= u_1, \quad \dot{u}_1 = u_4, \quad \dot{u}_2 = -u_3 u_5, \quad \dot{u}_3 = -2u_3 u_4, \\ \dot{u}_4 &= u_1 u_3 - u_3 u_5, \quad \dot{u}_5 = 2u_2 u_5 \end{aligned} \tag{57}$$

and the accompanying initial conditions

$$\begin{aligned} \xi(0) &= a, \quad u_1(0) = \ln\left(1 + a\right), \quad u_2(0) = \frac{1}{1+a}, \quad u_3(0) = \frac{1}{(1+a)^2}, \\ u_4(0) &= \frac{\ln\left(1+a\right)}{1+a}, \quad u_5(0) = \ln\left(1+a\right)^2 \end{aligned} \tag{58}$$

where we could discard the equations for ξ and its initial value since ξ can be solved from the expression of u_2. The equations have apparently conical structures. However, they are not unique since some terms can be equivalently written as second degree terms in some other unknowns. The singularities are not removed completely, although they are taken away from the descriptive functions. They appear in the initial conditions and they may not create problems unless a is taken equal to -1 or the expressions appearing there are attempted to be expanded in powers of a. The original ODE has a logarithmic branch point involving denumerably infinite number of Riemann sheets. Now the new descriptive functions are entire functions since they are multinomials. However, the initial conditions involve not only logarithmic branch point singularity but also a milder singularity, pole.

The descriptive functions may not be vanishing at any finite point. However, an extra space extension, as we have shown previously, takes us to a ten unknown ODE set whose descriptive function vanish at some set of points in the extended space.

We find this illustrative example sufficient to show the power of the space extension concept even though we have just focused on a one unknown case, since the dimensionality does not affect the main procedure.

10 Concluding remarks

In two papers, we have developed the probabilistic evolution based linearization of a set of explicit ODEs under certain initial conditions. The solutions are obtained through an infinite set of linear ODEs whose infinite coefficient matrix is a constant depending on the Taylor series coefficients of the descriptive functions (right hand side functions) of the ODEs. Even though the infinite equation concept has been investigated previously, the construction style is based completely on a mathematical concept basis function. We enumerate what we have obtained as original findings in this couple of papers below.

1. The construction is realized by using basis function expansion;
2. The importance of the triangularity and hypertriangularity in the evolution matrix and evolution folmat is evident and the ways to get this feature are revealed. Especially the space extension concept has proven to be a very efficient tool in this context;
3. The conicality in the descriptive functions gains a lot of importance in the techniques to be developed for the solutions. The truncated approximants have been defined and their construction is facilitated by conicality because of the first order recursive nature of the elementwise or block elementwise probabilistic equations;
4. The investigations have shown that the conicalization can be made possible by using space extension in many practical applications;
5. What we have developed here has been based on the assumption that the descriptive functions are entire functions. If they are not, then an appropriate space extension produces a set of new ODEs whose descriptive functions have entirety in extended space;
6. We have also discovered that the space extension can move the singularities in the descriptive functions to the initial conditions of the ODEs in the extended space;
7. We have revealed the role of the evolution operator in the convergence analysis for both univariate and multivariate cases;

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